Approximation algorithms for Art Gallery problems

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How many guards?
How many lights?  How many guards?

ART GALLERY Problem
Find \( g(n) \), the minimum number of guards that are always sufficient and sometimes necessary for guarding any polygon with \( n \) vertices.

\[ g(n) = \lfloor n/3 \rfloor \]

The problem is NP-hard. Aggarwal, 84. Lee-Lin, 86.

Minimize the number of guards for \( P \) \( g(P) \).

**COMBINATORIAL PROBLEM**

**ART GALLERY Problem**
1) Triangulate the polygon

2) Coloring the vertices with three colours

3) Place one guard at each vertex of the smallest chromatic class

\[ R + A + V = n \]
ART GALLERY Problem

1) Triangulate the polygon
2) Coloring the vertices with three colours
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Chvátal, 1975
Fisk, 1978

\[ R + A + V = n \]

\[ V \leq \left\lfloor \frac{n}{3} \right\rfloor \]
“Any polygon with $n$ vertices can be guarded with $\left\lfloor \frac{n}{3} \right\rfloor$ guards”
Theorem

\( \lfloor n/3 \rfloor \) guard are always sufficient and occasionally necessary to illuminate a polygonal art gallery with \( n \) vertices
VARIANTS

Which are the objects to guard?
• orthogonal polygons
• polygons with holes
• interior, walls, exterior …

How are the objects illuminated?
• from vertex (vertex-guards), interior points (guards)
• mobile guards (edge guards, diagonal guards)
• floodlights
• through the walls …
Minimizing the number of guards for a polygon P is NP-hard

- Simple polygons vertex guards, (interior) guards
  Lee-Lin, Aggarwal, 84
- Orthogonal polygons vertex guards, (interior) guards
  Schuchardt, Hecker, 95
Minimizing the number of guards for a polygon P is NP-hard

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  - vertex guards, (interior) guards
  - Lee-Lin, Aggarwal, 84

- Orthogonal polygons
  - vertex guards, (interior) guards
  - Schuchardt, Hecker, 95

- Polygons with link-diameter 2
  - Broden, Hammar, Nilsson, 01
ART GALLERY Problem

Minimizing the number of guards for a polygon \( P \)

There is an exact algorithm for **ONE** type of guarding

Orthogonal polygon  Rectangle visibility

Keil, Worman, 06  \( O(n^{17}) \)
How many edge-guards?

**COMBINATORIAL PROBLEM**

Conjecture (Toussaint, 1981)

\[ \lceil n/4 \rceil \text{ edge-guards are always sufficient and sometimes necessary for guarding any polygon with } n \text{ vertices} \]
How many edge-guards?  Conjecture (Toussaint, 1981)

**COMBINATORIAL PROBLEM**

\[ \lceil \frac{n}{4} \rceil \] edge-guards are always sufficient and sometimes necessary for watching any polygon with \( n \) edges

... except these polygons

necessary
ALGORITHMIC PROBLEM

Minimize the number of edge-guards for P

NP-hard
Lee, Lin, 84
Orthogonal polygons

Minimize the number of edge-guards for P

NP-hard
Orthogonal polygons

Minimizing the number of orthogonal vertex floodlights for a polygon P is NP-hard?
Any orthogonal polygon can always be illuminated with at most \( \frac{3n - 4}{8} \) orthogonal vertex floodlights. The bound is tight. Estivill, Urrutia, 94
If the signal can cross two walls, we need two routers in this polygon.

Minimizing the number of k-routers for covering a polygon P is NP-hard?

k-router can cross k walls
VISIBILITY THROUGH THE WALLS

Wireless router!!

COMBINATORIAL PROBLEM

2-routers

\[ \lceil \frac{n}{6} \rceil \ ?? \]

ALGORITHMIC PROBLEM

OPEN PROBLEM
How many guards?

Minimize is a NP-hard problem
Cole-Sharir, 89

VERTEX (POINT) GUARD
FIXED HEIGHT GUARD
How many guards?

**Vertex-guards**

\[ \left\lfloor \frac{n}{2} \right\rfloor \] are always sufficient and sometimes necessary
Bose, Shermer, Toussaint, Zhu, 92

**Edge-guards**

\[ \left\lfloor \frac{n}{3} \right\rfloor \] are always sufficient (Everett, Rivera-Campo, 94) and
\[ \left\lfloor \frac{(4n-4)}{13} \right\rfloor \] are sometimes necessary (BSTZ, 92, 97)
Watchtower placement problems

- Discrete version: bases at vertices of T
- Continuous version: bases anywhere on T

There are k watchtowers, we want to minimize their height

- Single watchtower: $O(n \log n)$ Zhu, 97
- Two watchtowers: Agarwal, Bereg, Ntafos, Zhu, 05
  - 1.5D discrete version: $O(n^2 \log^4 n)$
  - Continuous version: $O(n^3 \alpha(n) \log^3 n)$
Watchtower placement problems

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- Two watchtowers Agarwal, Bereg, Ntafos, Zhu, 05
  - 1.5D discrete version \( O(n^2 \log^4 n) \)
  - continuous version \( O(n^3 \alpha(n) \log^3 n) \)
  - 2.5D discrete version \( O(n^{11/3} \text{polylog}(n)) \)
An algorithm $A$ for solving a minimization problem with cost function $f$, is a $s$-approximation algorithm if

$$f(\text{solution } A) \leq s \cdot f(\text{optimum solution})$$

$s$ factor of approximation

Reduction to **SET COVER**

Visibility polygons of the vertices
The greedy algorithm for SET COVER produces a vertex-guard cover in $O(\log n)$-approximation time $O(n^5 \log n)$.

Efrat, Har-Peled, 02 (using randomization and VC-dimension)
- vertex-guard cover
  - If $c$ is the optimum, the approximation is $O(\log c)$-approximation time $O(nc^2 \log^4 n)$.
- guards in a grid
  - $O(\log c)$-approximation time $O(nc^2 \log c \log(nc) \log^2 \Delta)$

No approximation bounds for the greedy approach are known if guards can be located in the interior of the polygon.
ART GALLERY Problem

APPROXIMATION ALGORITHMS

Constant-factor approximations

Nilsson, 05
- monotone polygons (interior guards)
  12-approximation polynomial time
- orthogonal polygon (interior guards)
  96c-approximation

1.5D terrains (exploiting geometric structure of terrains)
- Ben-Moshe, Katz, Mitchell, 04
  O(1)-approximation time \(O(n^4)\)
- King, 06
  5-approximation time \(O(n^2)\) discrete and continuos
(In-)Approximability, Eidenbenz, 2000

- Polygons without holes
  \textbf{MINIMUM VERTEX (POINT) GUARD} is APX-hard
  There exists $\varepsilon > 0$ such that no polynomial time algorithm can achieve an approximation ratio of $1 + \varepsilon$

- Polygons with holes
  \textbf{MINIMUM VERTEX (POINT) GUARD} can not be approximated with an approximation ratio of $O(\log n)$

The problem is $O(\log n)$-complete for vertex-guard
ART GALLERY Problem

A ONE POINT GUARD
B ONE POINT GUARD (holes)
C MINIMUM VERTEX/EDGE GUARD
D MINIMUM VERTEX/EDGE GUARD (holes)
E MINIMUM POINT GUARD
F MINIMUM POINT GUARD (holes)
G MAX LENGTH/AREA/VALUE VERTEX/EDGE GUARD
H MAX LENGTH/AREA/VALUE VERTEX/EDGE GUARD (h)
I MAX LENGTH/AREA/VALUE POINT GUARD
J MAX LENGTH/AREA/VALUE POINT GUARD (h)
**ART GALLERY Problem**

**APPORXIMATION ALGORITHMS**

- Amit, Mitchell, Packer, '07
  Systematic experimentation with many guard placement heuristics

- Bottino, Laurentini, 04, 05, 06
  Optimal positioning of sensors in 2D
  Integer Edge Covering in polygons

- Bajuelos, Marques, Martins, Tomás, 04, 05, 06
  Vertex Guard problem for orthogonal polygons

- Canales, Abellanas, Alba, H., 04, 07
  Point Guard problem

- Bajuelos, Canales, H., Martins, 07, 08
  Vertex Guard problem
  Maximum Hidden Vertex Set problem

**Heuristics**
- Genetic algorithm
- Simulated annealing
An algorithm $A$ for solving an minimization problem with cost function $f$, is a $s$-approximation algorithm if

$$f(\text{solution } A) \leq s \cdot f(\text{optimum solution})$$

$s$ factor of approximation

.... but we don’t know the optimum solution!
Basic references for Art Gallery problems


Open problems

• The Open Problems Project (Demaine, Mitchell, O’Rourke)  
  http://maven.smith.edu/~orourke/TOPP/