Safe Routes on a Street Graph with Minimum Power Transmission Range

Manuel Abellanas∗†, Antonio L. Bajuelos‡§, Inês Matos‡¶

Abstract

Let $S$ be a set of $n$ points in the plane (antennas). An object is said to be 2-covered with range $r$ if every point of such object is interior to at least two discs (not necessarily the same) centered at $S$ of radius $r$. The following problem is considered in this paper: given a set $S$ of $n$ antennas and a planar geometric graph $G = (V, E)$, calculate the minimum power transmission range of $S$ so that a 2-covered path between two given nodes of $G$ exists. Is is described an algorithm to solve this problem in two phases. In the first phase (preprocessing phase), graph $G$ is transformed into a weighted graph $G_w$ (using the second order Voronoi diagram of $S$) and then a minimum spanning tree of $G_w$, $T_w$, is found. This phase takes $O(|E| \times n)$, $|E| > \log n$, time. In the second phase (solution phase), the minimum power transmission range of $S$ and a 2-covered path are calculated using $T_w$. Regarding time complexity, this second phase is linear on the number of edges of $T_w$.

1 Introduction and Related Works

Let $S$ be a set of $n$ points in the plane that represent the location of $n$ antennas (or any device able to send or receive some sort of signal). Suppose all antennas have the same power transmission range $r \in \mathbb{R}^+$ that is variable. The distance between a point $q$ and a set $S$ of points is the shortest distance between $q$ and every point of $S$. The minimum range of the antennas so that $S$ covers $q$ is exactly the distance between $q$ and $S$. A point covered by two or more antennas is said to be 2-covered by $S$. The concept of 2-covering is useful to assure that the points remain covered when one antenna fails. Let $D$ be the set of discs centered at the antennas of $S$ of radius $r$. Each nonempty intersection between two discs of $D$ is called a lens. It is easy to see that 2-covered regions result from the union of these lenses. Therefore, a point is 2-covered by $S$ if it belongs to a lens (see Figure 1). The antennas’ covering range, $r$, depends directly on their transmission power which, on its turn, is responsible for the costs associated to the service they provide. For that reason, it is important to minimize the value of $r$.

Consider now a connected geometric planar graph $G$ together with set $S$. Suppose that the edges of $G$ represent streets/roads and its nodes represent reachable locations using those streets (see Figure 1). A path on $G$ using only the edges that are 2-covered by $S$ is called a 2-covered path or a 2-path. Given two nodes $n_i$ and $n_j$ of $G$, our main goal is to compute the minimum power transmission range so that there is a 2-path on $G$ connecting $n_i$ and $n_j$.

Related Works: in [1] the authors study a problem equivalent to the one here proposed and other multi-parametric optimization problems for 1-covering. In [2], a subset of a given set of discs with variable radii whose costs depend on their radii is computed, as well as the same subset but with fixed costs to cover a given line segment at minimum cost.

This paper is structured as follows. The next section is divided in two subsections. In the first subsection, graph $G$ is preprocessed and converted into

∗Facultad de Informática, Universidad Politécnica de Madrid, mabellanas@fi.upm.es
†Co-supported by Project Consolider Ingenio 2010 i-MATH C3-0159 and MICINN Project MTM2008-05043
‡Departamento de Matemática & CEC, Universidade de Aveiro, {leslie.ipmatos}@ua.pt
§Supported by CEC through Programa POCTI, FCT, co-financed by EC fund FEDER
¶Supported by a FCT fellowship, grant SFRH/BD/28652/2006 and by CEC through Programa POCTI, FCT, co-financed by EC fund FEDER
a weighted graph $G_w$. In this subsection, a minimum spanning tree (MST) of $G_w$ is also found. Such MST is later used in the second subsection to calculate the minimum power transmission range of $S$ to guarantee the existence of a 2-path between two nodes of $G$. Finally, conclusions and some remarks are presented in Section 3.

2 Minimum 2-Covering of a Path between two nodes of a Street Graph

Let $S$ be a set of $n$ antennas. The second order Voronoi diagram of $S$, $VD_2(S)$, divides the plane into several regions by grouping points that share the same two closest antennas [3]. This proximity structure is naturally related to this problem since a point is 2-covered if it is interior to the lens defined by its two closest antennas. For this reason, a point $q$ is 2-covered if the power transmission range of the two closest antennas to $q$ is enough to reach $q$. The minimum power transmission range of $S$ that 2-covers an object $x$ is denoted by $MR_S(x)$.

Let $G = (N, E)$ be a connected geometric planar graph whose nodes represent locations and its edges represent roads/streets that connect such locations. Without loss of generality, it is assumed that $|E| > \log n$. Given two nodes $n_i$ and $n_j$ of $G$, the smallest range of $S$ which ensures the existence of a 2-path between $n_i$ and $n_j$ on $G$ is denoted by $MR_{G}(n_i, n_j)$.

To simplify the notation, it can also be denoted by $MR_S(n_i, n_j)$ if $G$ is clear from the context.

2.1 First Phase: Preprocess

![Figure 2: The minimum power transmission range $r$ of $S$ to 2-cover $e_1\overrightarrow{e_2}$ is $r = d(e, s_3) = d(e, s_4)$. $VD_2(S)$ is shown in a dashed trace.](image)

Given a line segment $e$, $MR_S(e)$ is calculated using the intersection points between $e$ and $VD_2(S)$, $I_e = \{e\} \cap VD_2(S)$. It is easy to see that the minimum power transmission range needed to 2-cover every point of $I_e$ and the extreme points of $e$ is $MR_S(e)$ (see Figure 2).

The procedure explained in this subsection acts as a preprocess that transforms graph $G$ into weighted graph $G_w$. To make this transformation, $MR_S(e)$ is assigned as the weight of each edge $e$ of $G$ (see Figure 3).

**Proposition 1** Let $G = (N, E)$ be a graph and $S$ a set of $n$ antennas. The $MR_S(e)$ for every edge $e \in E$ can be calculated in $O(|E| \times n)$ time.

**Proof.** Computing $VD_2(S)$ takes $O(n \log n)$ since there are $n$ antennas [6]. Then using $VD_2(S)$, it is possible to find the set $I_e = \{e\} \cap VD_2(S)$ for every edge $e \in E$ in $O(n \log n + |E| \times n)$ time. Next there is the need to add the extreme points of $e$ to the set $I_e$. For each intersection point $p \in I_e$, $MR_S(p)$ is calculated in constant time using $VD_2(S)$ as it is the distance between $p$ and its second closest antenna. Therefore, calculating $MR_S(e) = \max\{MR_S(p), \forall p \in I_e\}$ is a lin-

![Figure 3: Set $S$ of eleven antennas represented by dots. $VD_2(S)$ is shown in a dashed trace. (a) Graph $G$ whose nodes are represented by squares. (b) The graph’s edge connecting $n_1$ and $n_7$ has weight 38 which is the minimum power transmission range of $S$ to 2-cover such edge.](image)
ear procedure for each edge $e$. As it is supposed that $|E| > \log n$, $MR_S(e)$ for every edge $e \in E$ can be computed in $O(|E| \times n)$ time. □

Let us remark that a minimum spanning tree of $G_w$, $T_w$, can be found in linear time using an algorithm by Matsui [4] since $G_w$ is a planar graph. As it is shown in the next subsection, $T_w$ eases the calculations since it stores important information concerning the antennas’ arrangement in relation to $G_w$. To summarize, the algorithm can be described as follows.

**Algorithm: Preprocessing Phase**

**In:** Set $S$ of $n$ antennas and a street graph $G$

**Out:** Graph $G_w$ and $T_w$, a MST of $G_w$

1. Compute $VD_2(S)$, the Second Order Voronoi Diagram of $S$.
2. Convert $G$ into weighted graph $G_w$:
   For every $e \in G$ do
   (a) Calculate $MR_S(e)$
   (b) $w(e) \leftarrow MR_S(e)$
3. Find $T_w$, a MST of $G_w$.

The next result is a direct consequence of Proposition 1.

**Theorem 2** Given a set $S$ of $n$ antennas and graph $G = (N, E)$, computing a weighted graph $G_w$ and a MST of $G_w$ can be done in $O(|E| \times n)$ time.

### 2.2 Second Phase: Solution

Given two nodes $n_i$ and $n_j$ of $G_w$, the algorithm in this subsection shows how to calculate $MR_{S,G_w}(n_i, n_j)$ and how to find a 2-covered path on $G_w$ connecting $n_i$ and $n_j$.

Let $P$ be a 2-covered path between two nodes on $G_w$. It is easy to see that the minimum power transmission range that ensures the existence of $P$ is given by the weight of the heaviest edge of $P$. The next property shows that it is only necessary to consider the edges of a MST of $G_w$ to find a 2-path between two given nodes of $G_w$.

**Proposition 3** Let $G_w$ be an edge-weighted connected graph. For any path $P$ on $G_w$, let us consider the weight of $P$ as the weight of its heaviest edge. Then, for every pair of nodes of $G_w$, the path connecting them on a MST of $G_w$ is a minimum weight path between such pair.

**Proof.** Let $G_w$ be an edge-weighted connected graph and $T_w$ a minimum spanning tree (MST) of $G_w$. Suppose that $P$ is the only path on $T_w$ connecting nodes $n_i$ and $n_j$ and $e$ is its heaviest edge. Consequently, $P$ has weight $w(e)$. Now suppose that path $P^*$ on $G_w$ is a minimum weight path connecting $n_i$ and $n_j$. Its weight is given by $e^*$, its heaviest edge, and so $w(e^*) < w(e)$. Since $P$ is heavier than $P^*$, the edge $e$ is not part of $P^*$. If paths $P$ and $P^*$ are united, then a cycle is created. That cycle contains $e$ which clearly is its heaviest edge. But this contradicts the hypothesis that $e$ is an edge of a MST of $G_w$. Therefore, a minimum weight path between two nodes of $G_w$ is the path on $T_w$ connecting those nodes. □

![Figure 4](image-url)

Figure 4: (a) A MST of a weighted graph is shown in a solid trace. (b) The path connecting $n_1$ and $n_8$ on the tree only exists if the antennas’ power transmission range is at least $d(s_8, n_1) = d(s_{10}, n_1) = 38$.

It is now clear that a 2-path between two nodes on a weighted graph with minimum power transmission range can be computed using a MST of such graph (see Figure 4(a)). Working with a MST is easier since a path between two nodes is unique. Follows the core of the algorithm to find a 2-path connecting two nodes of $G_w$ with minimum range, taking advantage of a...
MST of $G_w$.

**Algorithm: Solution Phase**

**In:** $T_w$, a MST of $G_w$

**Out:** $P$, a 2-covered path and $MR_{S,G_w}(n_i, n_j)$

1. Use the algorithm DFS [5] to find a path $P$ on $T_w$ between $n_i$ and $n_j$.
2. $MR_{S,G_w}(n_i, n_j) \leftarrow \max\{w(e) : \forall e \in P\}$.

In Figure 4(b) there is an example of a path on a MST connecting nodes $n_1$ and $n_8$ with minimum power transmission range. If all the antennas have range $r = \max\{31, 32, 28, 38\} = 38$, the path is 2-covered with minimum power transmission range. The largest range is the one needed to 2-cover $n_1$. It is given by the distance between $n_1$ and $s_8$ (or $s_{16}$), such distance is exactly $MR_{S,G_w}(n_1, n_8)$.

**Theorem 4** Given a set $S$ of $n$ antennas, let $n_i$ and $n_j$ be two nodes of graph $G_w$ and $T_w = (N, B)$ a MST of $G_w$. Then a 2-covered path between $n_i$ and $n_j$ on $G_w$ with range $MR_{S,G_w}(n_i, n_j)$ can be computed on $T_w$ in $O(|B|)$ time.

3 Concluding Remarks

The concept of 2-covering was introduced in this paper, as well as a related problem: calculating the minimum power transmission range to ensure the existence of a 2-covered path between two nodes of a graph. Given a set $S$ of $n$ antennas, the solution to this problem is divided in two phases. In the first phase, graph $G = (N, E)$ is converted into a weighted graph using a $O(|E| \times n)$, $|E| > \log n$, time algorithm. In the second phase, a 2-path between two nodes of $G$ is found, as well as the minimum power transmission range of $S$ to guarantee the existence of such path. This last algorithm is linear on the number of edges of a minimum spanning tree of the weighted graph.

References


